spin $\frac{1}{2}$ - Pauli Theory:

The spin angular momentum theory is similar to the orbital angular momentum theory. the spin of a particle is a vector operator $\hat{\vec{S}} = \hat{S}_x \hat{i} + \hat{S}_y \hat{j} + \hat{S}_z \hat{k}$; $[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$

let us denote the spin eigenstates by $|sm\rangle$, so the spin eigenvalue equations are (choosing the z-axis as our polar axis)

$$S^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle \quad \text{and} \quad S_z |sm\rangle = \hbar m_s |sm\rangle$$

- Consider a spin $1/2$ particle. the projection of s onto the z-axis can only takes two values $m = 1/2$ or $m = -1/2$

Conversions and notations

$$|1/2, 1/2\rangle = \chi_{1/2} = \chi_+ = \alpha = \uparrow \quad (s = 1/2,\ m = 1/2)$$

$$\underbrace{|1/2, -1/2\rangle = \chi_{-1/2} = \chi_- = \beta = \downarrow}_{\text{called spinors}} \quad (s = 1/2,\ m = -1/2)$$

or mostly $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

these spinors are normalized and form a complete set

$$\chi_+ \chi_+^\dagger = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \quad 0) = 1 \quad \text{and} \quad \chi_- \chi_-^\dagger = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \quad 1) = 1$$

and orthogonal $\chi_+ \chi_-^\dagger = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \quad 1) = 0$,

since these bases for a complete set of orthogonal states, then any general spin state can be expanded into these basis

$$\chi = a\chi_+ + b\chi_- , \text{ where } |a|^2 + |b|^2 = 1$$

$$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

the raising and lowering operators $S_\pm$ for spin is defined as $S_\pm = S_x \pm i S_y \Rightarrow$ using $S_\pm \underbrace{|s,m\rangle} = \hbar\sqrt{s(s+1) - (m\pm 1)m}\ |s, m\pm 1\rangle$

one finds $S_+ |1/2, 1/2\rangle = S_+ |\chi_+\rangle = 0$

$$= \hbar\sqrt{1/2(1/2+1) - 1/2(1/2+1)}\ |1/2, \tfrac{1}{2}+1\rangle = 0$$

also $S_- |1/2, -1/2\rangle = S_- |\chi_-\rangle = 0$

and $S_+|\chi_-\rangle = \hbar|\chi_+\rangle$ and $S_-|\chi_+\rangle = \hbar|\chi_-\rangle$

Now all the spin operators S^2, S_z, S_+, S_- can be represented as 2×2 matrices.

- (S_z) $\quad S_z|\chi_+\rangle = \frac{\hbar}{2}|\chi_+\rangle$ so $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow a = \frac{\hbar}{2} \text{ and } c = 0$$

$S_z|\chi_-\rangle = -\frac{\hbar}{2}|\chi_-\rangle \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow b = 0 \text{ and } d = -\frac{\hbar}{2}$$

$$\Rightarrow S_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\sigma_z \quad ; \text{ where } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrix

- S_+ $\quad S_+|\chi_+\rangle = 0 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \Rightarrow$ $a = 0$, $c = 0$

$S_+|\chi_-\rangle = \hbar|\chi_-\rangle \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$ $b = 1$, $d = 0$

$$\Rightarrow S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

similarly for $S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Now one can calculate S_x and S_y

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \quad ; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y \quad ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

so in general $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$; $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

Notice that χ_+, χ_- are eigenstates for S^2 and S_z but Not for S_x and S_y

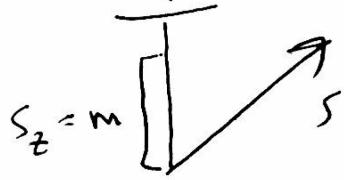
$$S_x|\chi_+\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\chi_-\rangle$$

so χ_+ is Not an eigen state for S_x

but $$S_x^2|\chi_+\rangle = \frac{\hbar}{2}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}|\chi_+\rangle$$

so χ_+ is an eigen value of the operator $\underline{S_x^2}$

$$\hat{\vec{S}} = \hat{S}_x\hat{i} + \hat{S}_y\hat{j} + \hat{S}_z\hat{k}$$



S_z is certain $\langle S_z\rangle = m$

S_x, S_y are Not certain $\langle S_x\rangle = \langle S_y\rangle = 0$

proof: $\langle S_x\rangle = \langle\chi_+|S_x|\chi_+\rangle = \langle\chi_-|S_x|\chi_-\rangle$

$$= (1\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

This is the average value of measuring S_x along the x-axis

This is expected as there is No polarization along the x-axis.

General properties of Pauli matrices:

$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k \Rightarrow$ using $S_i = \frac{\hbar}{2}\sigma_i$

$\Rightarrow \frac{\hbar}{2}\frac{\hbar}{2}[\sigma_i, \sigma_j] = i\hbar \varepsilon_{ijk} \frac{\hbar}{2}\sigma_k$

$\Rightarrow [\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$

- $\sigma_i^{\dagger} = \sigma_i \quad ; \quad i = 1,2,3$
- $\det(\sigma_i) = 1$
- $(\sigma_i)^2 = (\sigma_j)^2 = (\sigma_k)^2 = \mathbb{1}$ identity $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\mathrm{tr}(\sigma_i) = 0 \quad ; \quad i = 1,2,3$
- $\sigma_i \sigma_j = -\sigma_j \sigma_i$ or $\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad ; \quad i,j = 1,2,3$
- $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k$
- $(\vec{a}.\vec{\sigma})(\vec{b}.\vec{\sigma}) = (\vec{a}.\vec{b})\mathbb{1} + i(\vec{a} \times \vec{b}).\vec{\sigma}$

for example take $\vec{a} = \vec{b} \Rightarrow \vec{a}.\vec{a} = |a|^2 \; ; \; \vec{a} \times \vec{a} = 0$

$\Rightarrow (\vec{a}.\vec{\sigma})^2 = |a|^2 \mathbb{1}$

when $\vec{a}$ is a unit vector $\vec{n} \Rightarrow (\vec{n}.\vec{\sigma})^2 = \mathbb{1}$

- the ε symbol satisfies $\varepsilon_{ijk}\varepsilon_{ijq} = 2\delta_{kq}$

and $\varepsilon_{ijk}\varepsilon_{ip\ell} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

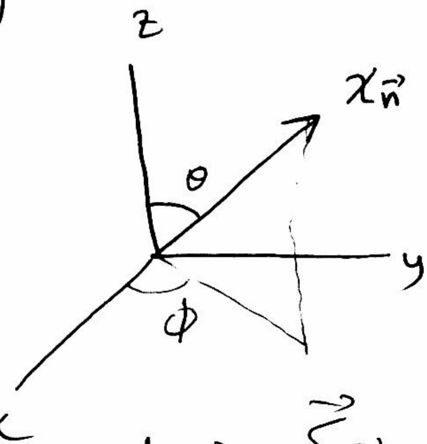
also one needs $a \times (b \times c) = b(a.c) - (a.b)c$

$e^{i\alpha\sigma_i} = \hat{I}\cos\alpha + i\sigma_i \sin\alpha \quad ; \quad i = 1,2,3$

Construction of an arbitrary spin state

consider a spin half-particle that points in arbitrary direction as specified by a unit vector $\vec{n}$

$\vec{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



θ, ϕ : polar and azimuthal angles

$\Rightarrow$ the spin of the particle is $\vec{S}$

$$\hat{\vec{S}} = \hat{S}_x \hat{i} + \hat{S}_y \hat{j} + \hat{S}_z \hat{k}$$
$$= \frac{\hbar}{2} \vec{\sigma}$$

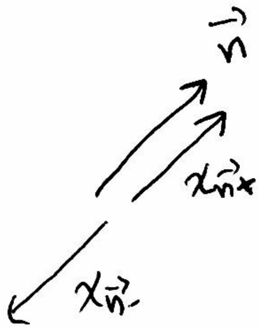
for specific direction, the spin of the particle is $\vec{S}_{\vec{n}}$

$$\vec{S}_{\vec{n}} = \vec{S}\cdot\vec{n} = \frac{\hbar}{2} \vec{\sigma}\cdot\vec{n}$$; Recall that $(\vec{\sigma}\cdot\vec{n})^2 = 1$

when $\vec{S}_n$ points in the direction of the unit vector $\vec{n}$

the eigenvalues of $\vec{S}_{\vec{n}}$ are $\pm\frac{\hbar}{2}$ and the eigenstates $\chi_{\vec{n}\pm}$

where $S_{\vec{n}} |\chi_{\vec{n}\pm}\rangle = \pm\frac{\hbar}{2} |\chi_{\vec{n}\pm}\rangle$



$|\chi_{\vec{n}+}\rangle$: the spin state that points up along $\vec{n}$

$|\chi_{\vec{n}-}\rangle$: " " " " down along $\vec{n}$

one can find the eigenvalue of $\vec{S}_n$ by

$$\vec{S}_n = \vec{S}\cdot\vec{n} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$
$$= \frac{\hbar}{2}\left(\sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z \cos\theta\right)$$
$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} \left(\frac{\hbar}{2}\cos\theta - \lambda\right) & \frac{\hbar}{2}\sin\theta\, e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta\, e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

as expected

Now to find the eigen states associated with $\pm\frac{\hbar}{2}$, we proceed as follow

$\chi_{\vec{n}+}$ state: this state can be constructed from χ_+ by rotating the χ_+ (along z-axis) by θ around the y-axis then followed rotation of χ_+ by ϕ around the z axis by an angle ϕ

$$\chi_{\vec{n}+} = R_z(\phi)\, R_y(\theta)\, \chi_+ = e^{-i\frac{\phi}{2}\sigma_z}\, e^{-i\frac{\theta}{2}\sigma_y} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now using the identity

$$e^{-i\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})} = 1\cos\left(\frac{\alpha}{2}\right) - i(\vec{\sigma}\cdot\vec{n})\sin\left(\frac{\alpha}{2}\right) \;;\; 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \chi_{\vec{n}+} = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \\ e^{+i\frac{\phi}{2}}\sin(\theta/2) \end{pmatrix} = e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\,\chi_+ + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\,\chi_-$$

Similarly one can construct the $\chi_{\vec{n}-}$ from χ_- by

$$\chi_{\vec{n}-} = R_z(\phi)\, R_y(\theta)\, \chi_- = e^{-i\frac{\phi}{2}\sigma_z}\, e^{-i\frac{\theta}{2}\sigma_y}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} = -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\,\chi_+ + e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\,\chi_-$$

to check

- take $\vec{n} = n_z \hat{k}$ spin points along the z-axis $\theta = 0, \phi = 0$

$$\Rightarrow \chi_{\vec{n}+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_+ \quad \text{and} \quad \chi_{\vec{n}-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_- \quad \text{as expected}$$

- take $\vec{n} = n_x \hat{i}$ spin points along the x-axis $\theta = \frac{\pi}{2}, \phi = 0$

$$\chi_{\vec{n}+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \chi_+ + \frac{1}{\sqrt{2}} \chi_-$$

$$\chi_{\vec{n}-} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \chi_+ + \frac{1}{\sqrt{2}} \chi_-$$

- take $\vec{n} = n_y \hat{j}$ spin points along the y-axis $\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}$

$$\chi_{\vec{n}+} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\pi/4} \\ \frac{1}{\sqrt{2}} e^{i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4} \end{pmatrix} \quad ; \quad e^{-i\frac{\pi}{4}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1-i)$$

$$e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$$

$$= \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$$

and

$$\chi_{\vec{n}-} = \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} \\ \frac{1}{\sqrt{2}} e^{i\pi/4} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1+i \\ 1+i \end{pmatrix}$$

$\chi_{\vec{n}+}$ and $\chi_{\vec{n}-}$ are always orthonormal

check for the last case $n = n_y \hat{j}$

$$\chi_{\vec{n}+} \chi_{\vec{n}+}^{\dagger} = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+i & 1-i \end{pmatrix} = \frac{1}{4}(2+2) = 1$$

and

$$\chi_{\vec{n}+} \chi_{\vec{n}-}^{\dagger} = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1-i & 1-i \end{pmatrix}$$

$$= \frac{1}{4}(-2+2) = 0$$

Example: a beam was prepared to be polarized in the (θ, ϕ) direction. the beam is then directed into an analyzer that measure the spin along the x-axis.

a) Find the probability of measuring $+\hbar/2$

$$\chi_i = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{+i\frac{\phi}{2}} \sin\theta/2 \end{pmatrix} \; ; \quad \chi_f = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P_+ = |\langle \chi_f | \chi_i \rangle|^2 = \left| \frac{1}{\sqrt{2}} (1 \;\; 1) \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\theta/2 \end{pmatrix} \right|^2$$

$$= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2} \underbrace{(e^{-i\phi} + e^{i\phi})}_{2\cos\phi} + \sin^2\frac{\theta}{2} \right)$$

$\sin 2x = 2\sin x \cos x$

$$= \frac{1}{2}\left(1 + \frac{1}{\cancel{2}} \sin\theta \; \cancel{2}\cos\phi\right)$$

$$= \frac{1}{2}(1 + \sin\theta \cos\phi)$$

b) find $\langle S_x \rangle$

$$\langle S_x \rangle = \langle \chi_i | S_x | \chi_i \rangle = \left(e^{i\frac{\phi}{2}} \cos\frac{\theta}{2} \quad e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2}\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\theta \cos\phi$$

c) $\vec{n} = n_x \hat{i}$ => original beam is polarized along x-axis

=> $\theta = \frac{\pi}{2}$, $\phi = 0$

=> $P_+ = 1$ and $\langle S_x \rangle = \frac{\hbar}{2}$ as expected

all beam pass through